On the nonasymptotic prime number distribution*

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Abstract

The objective of this paper is to introduce an approach to the study of the nonasymptotic distribution of prime numbers. The natural numbers are represented by theorem 1 in the matrix form ^2N. The first column of the infinite matrix ^2N starts with the unit and contains all composite numbers in ascending order. The infinite rows of this matrix except for the first elements contain prime numbers only, which are determined by an uniform recurrence law. At least one of the elements of the twin pairs of prime numbers is an element of the second column of the matrix ^2N (theorem 3). The basic information on the nonasymptotic prime number distribution is contained in the distribution of the elements of the second column of the matrix ^2N.

1 Introduction

The multiplicative and the additive structure of natural numbers is based on prime numbers. The derivation of results on the distribution of prime numbers in the set of natural numbers can affect almost all mathematical theories and their applications. The objective of this paper is to introduce an approach to the study of the nonasymptotic distribution of prime numbers. The results obtained could be applicable in quantum physics, quantum chemistry and molecular biology.

The Euclid theorem on the existence of an infinite number of prime numbers

$$\pi(x) \to \infty \qquad as\ x \to \infty, \tag{1}$$

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(where $\pi(x)$ is the number of primes not exceeding x), the Eratosthenes sieve formula

$$\pi(x) = \pi(\sqrt{x}) - 1 + \sum_d (-1)^{\nu(d)} [\frac{x}{d}], \tag{2}$$

(where d runs over the divisors of the products of all primes not exceeding $\sqrt{x}$, $\nu(d)$ is the number of the prime divisors of d and $[u]$ is the integer part of u) as well as the asymptotic distribution law of prime numbers (see f.i. [1] - [4])

$$\pi(x) = li\ x + O(xe^{-c\sqrt{\ln x}}) \quad as\ x \to \infty,\ c = const > 1, \tag{3}$$

where

$$li\ x = \int_2^x \frac{dt}{\ln t} = \frac{x}{\ln x} + \frac{1!x}{\ln^2 x} + \ldots + \frac{(k-1)!x}{\ln^k x} + O(\frac{x}{\ln^{k-1} x}) \tag{4}$$

do not answer the question how often the prime numbers are encountered and how they are distributed amidst the natural numbers when $x < \infty$. Even if the Riemann fifth hypothesis [5] stating that the nontrivial solutions of the equation

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} = 0, \quad s = \sigma + it \tag{5}$$

lie on the complex straight line ($\sigma = 1/2,\ t$) was proved it could not give an answer to that question too. The reason is that the fourth Riemann hypothesis [5] (proved by H. von Mangoldt) stating that the expression

$$P_0(x) = li\ x - \sum_\rho li\ x^\rho + \int_x^\infty \frac{du}{(u^2-1)\ln u} - \ln 2$$

where ρ runs over all nontrivial solutions of eq. (5) and

$$P_0(x) = \frac{1}{2}(P(x+0) + P(x-0)), \quad P(x) = \sum_{n \le x} \frac{\Lambda(n)}{\ln x} \ ,$$

$$\Lambda(n) = \begin{cases} \ln \rho, & when\ n = p^m \quad and\ p\ is\ a\ prime, \quad m \ge 1 \\ 0 & otherwise \end{cases}$$

only gives a connection between the distribution of the nontrivial zeroes of eq. (5) and the problem of the prime numbers distribution.

In the present paper an approach based on the separation of subsets of prime numbers for which explicit distribution law exists is used in the search for an answer to the question is there any exact law for $\pi(x)$, $2 \le x < \infty$ or the prime numbers are spread amidst the natural numbers in total disorder. The separation of these subsets

is made by means of a new sieve ("the multiple Eratosthenes sieve" [6]). The sieve and its generalizations will be published in a separate paper. In the presentation of the results on the nonasymptotic distribution of prime numbers the sieve is present only implicitly which simplifies the text. In the present paper all results are obtained by elementary methods except the use of the inequality from [7].

2 The prime number inner distribution law

Let the set of prime numbers is enumerated in incremental order of elements according to (1)

$$P = \{2, 3, 5, 7, 11, 13, \dots\} = \{p_n\}_{n \in N},$$

where N is the set of natural numbers. Thus two reciprocal number-theoretic functions are introduced

$$\pi(p) : P \to N \quad (i.e. \ \ \pi(p_n) = n)$$

"the number $\pi(p)$ of the prime number p" and

$$\pi^{-1}(n) : N \to P \quad (i.e. \ \ \pi^{-1}(n) = p_n)$$

"the prime number $\pi^{-1}(n)$ of number n".

It is evident that $\pi(p)$ and $\pi^{-1}(n)$ satisfy the identities

$$\pi(\pi^{-1}(n)) = n \tag{6}$$

and

$$\pi^{-1}(\pi(p)) = p.$$

These functions are also strictly monotonous

$$p' < p'' \Longrightarrow \pi(p') < \pi(p''), \quad p', p'' \in P,$$

$$n' < n'' \Longrightarrow \pi^{-1}(n') < \pi^{-1}(n''), \quad n', n'' \in N. \tag{7}$$

The following auxiliary proposition is a corollary from the inequality $p_n > n \ln n$ from [7].

Lemma 1 *For any prime number $n \geq 2$ the following estimates are correct:*

$$n > \pi(n) \ln \pi(n) \tag{8}$$

and

$$n < \frac{\pi^{-1}(n)}{\ln n}. \tag{9}$$

Using the function $\pi^{-1}(n)$ two basic entities of the present paper are introduced: the sequence ϵ_{p_0} and the aggregate r_{p_0}.

Definition 1 *The prime number sequence*

$$\epsilon_{p_0} : p_0 \in N, \quad p_{k+1} = \pi^{-1}(p_k), \quad k = 0, 1, 2, \dots \ , \tag{10}$$

is called "Eratosthenes progression of base p_0". The infinite prime number aggregate

$$r_{p_0} = \{p_{k+1} : p_0 \in N, \quad p_{k+1} = \pi^{-1}(p_k), \quad k = 0, 1, 2, \dots\} \tag{11}$$

is called "Eratosthenes ray of base p_0".

The following two auxiliary propositions are true for the Eratosthenes rays:

Lemma 2 *Let two rays $r_{p'_0}$ and $r_{p''_0}$ of bases*

$$p'_0 < p''_0 \tag{12}$$

are given. Then if:

1. *$p''_0 \in r'_{p_0}$ the ray $r_{p''_0}$ is contained in the ray $r_{p'_0}$;*
2. *$p''_0 \overline{\in} r_{p'_0}$ the ray $r_{p''_0}$ does not contain common elements with the ray $r_{p'_0}$ and the inequality (12) implies the inequalities*

$$p'_k < p''_k, \quad k = 0, 1, 2, \dots \ . \tag{13}$$

Proof

1. The inequality (12) gives the possibility to make the assumption that $p''_0 \in r_{p'_0}$. That means that a number $k^* > 0$ exists so that $p''_0 \equiv p'_{k^*}$. Now the inclusion $r_{p''_0} \subset r_{p'_0}$ follows from the recursion $p''_k = \pi^{-1}(p'_{k+k^*}), \quad k = 0, 1, 2, \dots$, given in (11).
2. Let $p''_0 \overline{\in} r_{p'_0}$. Now suppose a number $k^* > 0$ exists so that $p''_{k^*} \in r_{p'_0}$. This implies the existence of a finite reverse sequence

$$p''_k = \pi^{-1}(p''_{k+1}) \in r_{p'_0}, \quad where \ \ k = k^* - 1, \ k^* - 2, \ k^* - 3, \dots, 0,$$

that terminates on the inclusion $p''_0 \in r'_{p_0}$ which contradicts the initial assumption $p''_0 \overline{\in} r_{p'_0}$. Thus the ray $r_{p''_0}$ does not have common elements with the ray $r_{p'_0}$.

The inequality (12) gives the possibility to use the implication (7) together with the recursion (11) that defines the elements of $r_{p_0'}$ and $r_{p_0''}$ and this leads to the following chain of implications

$$\begin{aligned} p_0' < p_0'' &\implies p_1' = \pi^{-1}(p_0') < p_1'' = \pi^{-1}(p_0''), \\ p_1' < p_1'' &\implies p_2' = \pi^{-1}(p_1') < p_2'' = \pi^{-1}(p_1''), \\ &\vdots \\ p_{k-1}' < p_{k-1}'' &\implies p_k' = \pi^{-1}(p_{k-1}') < p_k'' = \pi^{-1}(p_k''). \quad [] \end{aligned}$$

Lemma 3 *The principal Eratosthenes rays do not intersect*

$$\bigcap_{p_0 \in \overline{C}} r_{p_0} = \emptyset \tag{14}$$

where

$$\overline{C} \equiv N \backslash P = \{1, 4, 6, 8, 9, 10, 12, \ldots\}.$$

The union of all principal rays contains the union of all rays

$$\bigcup_{p_0 \in P} r_{p_0} \subset \bigcup_{p_0 \in \overline{C}} r_{p_0}. \tag{15}$$

Proof

The proving method is by induction.
The first induction step starts considering the following 6 cases:

- *Case* $p_0 = 1 \in \overline{C}$. The progression ϵ_1 from (10) generates the first principal prime numbers ray r_1;
- *Case* $p_0 = 2 \in r_1$ *and* $p_0 = 3 \in r_1$. The progressions ϵ_2 and ϵ_3 from (10) determine the rays r_2 and r_3, which according to lemma 2(1) do not contain any new elements compared to the principal ray that is generated already. So the following inclusion is true:

$$r_2 \bigcup r_3 \subset r_1; \tag{16}$$

- *Case* $p_0 = 4 \in \overline{C}$. The progression ϵ_4 from (10) generates the second principal prime numbers ray r_4, which according to lemma 2(2) does not have common elements with the ray r_1. Thus their intersection is void:

$$r_1 \bigcap r_4 = \emptyset; \tag{17}$$

- *Case* $p_0 = 5 \in r_1$. According to lemma 2(1) the Eratosthenes progression ϵ_5 generates the ray $r_5 \subset r_1$. Together with (16) this inclusion implies the new inclusion:
$$r_2 \bigcup r_3 \bigcup r_5 \subset r_1; \tag{18}$$
- *Case* $p_0 = 6 \in \overline{C}$. According to lemma 2(2) the Eratosthenes progression ϵ_6 generates the third principal prime numbers ray, which does not contain common elements with the already existing principal rays r_1 and r_4. Together with the relation (17) this leads to a new particular case of the equality (14)
$$r_1 \bigcap r_4 \bigcap r_6 = \emptyset;$$
- *Case* $p_0 = 7 \in r_4$. According to lemma 2(1) the Eratosthenes progression ϵ_7 generates the ray $r_7 \subset r_4$. This inclusion together with the inclusion (18) imply the extended inclusion
$$(r_2 \bigcup r_3 \bigcup r_5 \bigcup r_7) \subset (r_1 \bigcup r_4),$$
On its turn this inclusion appears to be a particular case of the inclusion (15).

The second induction step. Suppose the relations
$$\bigcap_{p_0 \in C_n} r_{p_0} = \emptyset \ , \tag{19}$$
$$R_n \equiv \bigcup_{p_0 \in P_{c_n}} r_{p_0} \subset \bigcup_{p_0 \in C_n} r_{p_0} \ , \tag{20}$$
where
$$C_n = \bigcup_{i=1,2,3,\ldots,n} c_i, \quad P_{c_n} = \{p \in P : \ p < c_n\},$$
are satisfied up to the n-th arbitrary element $c_n > 7$ of the set $\overline{C}$.

We shall prove that the relations (19) and (20) remain true for $c_n = c_n + 1$. Let us consider the following two cases:

- *Case* $p_0 = c_n + 1 \in P$. We shall proceed starting from the inequality
$$c_{n+1} > \pi(c_n + 1) \tag{21}$$
(the prime number c_{n+1} is greater than its ordinal number $\pi(c_n + 1)$), which follows directly from inequality (8).

The right hand side of inequality (21) $\pi(c_n+1)$ is either a composite or a prime number. Assume that $\pi(c_n+1)$ is a prime number. Then applying the operation π on both sides of (21) we obtain a double chain of inequalities

$$c_{n+1} > \pi(c_n+1) > \pi^2(c_n+1). \tag{22}$$

Now we assume that the "second number" $\pi^2(c_n+1)$ is again a prime number and apply once more the operation π on every element of the double chain of inequalities (22). Thus we obtain a triple chain of inequalities analogous to (22). The process can go further α-times obtaining α-multiple chain of inequalities

$$c_{n+1} > \pi(c_n+1) > \pi^2(c_n+1) > \ldots > \pi^\alpha(c_n+1) \tag{23}$$

until the "α-number" $\pi^\alpha(c_n+1) \equiv p_0^* \in \overline{C}$.

Applying the operation π on both sides of the recurrence formula (11) for the ray $\epsilon_{p_0^*}$ and taking into account the identity (6) we obtain the generator of the finite reverse sequence

$$p_i^* = \pi(p_{i+1}^*), \quad i = \alpha,\ \alpha-1,\ \alpha-2,\ \ldots, 0. \tag{24}$$

The terms of the inequalities (23) are generated by the (24): $c_n+1 = p_\alpha^*$, $\pi(c_n+1) = p_{\alpha-1}^*$, $\pi^2(c^n+1) = p_{\alpha-2}^*$, $\ldots$, $\pi^\alpha(c_n+1) = p_0^*$. Thus the inequality $c_n+1 > \pi^\alpha(c_n+1)$ and the fact that c_n+1 belongs to the principal ray of base $\pi^\alpha(c_n+1)$ (the latter is shown by the sequence (24)) allows for the use of the lemma 2(1). This implies the inclusion $r_{c_n+1} \subset R_n$, which together with eq. (20) allows for the needed extension of the inclusion (20)

$$r_{c_n+1} \bigcup \left(\bigcup_{p_0 \in P_{c_n}} r_{p_0} \right) = \bigcup_{p_0 \in P_{c_n+1}} r_{p_0} \subset \bigcup_{p_0 \in C_n} r_{p_0} .$$

- *Case* $c_n+1 = c_{n+1} \in \overline{C}$. According to lemma 2(2) the Eratosthenes progression $\epsilon_{c_{n+1}}$ generates the new principal ray $r_{c_{n+1}}$, which is not contained in the union R_n. So that due to the extension of relation (19)

$$\bigcap_{p_0 \in C_{n+1}} r_{p_0} = \emptyset .$$

The third induction step takes into account the limits $C_n \longrightarrow \overline{C}$ for $n \longrightarrow \infty$ and $P_{c_n+1} \longrightarrow P$ for $n \longrightarrow \infty$. Thus we find out that the relations (19) and (20) in

that limit go to the relations (14) and (15) respectively. []

Lemma 3 makes possible to show that the following basic theorem concerning the separating of prime numbers into subsets with explicit law for the determination of their elements actually takes place (Prime Number Separating Theorem - PNST [6]).

Theorem 1 *The set of prime numbers has a two-dimensional representation labeled by the index k and the base p_0*

$$P = \bigcup_{p_0 \in N \setminus P} \left\{ r_{p_0} = \{p_{k+1} : \ p_{k+1} = \pi^{-1}(p_k), \ k = 0, 1, 2, \ldots\} \right\}. \tag{25}$$

Proof

The union of all Eratosthenes rays of bases covering the set of all natural numbers

$$Q = \bigcup_{p_0 \in N} \{p_{k+1} : p_{k+1} = \pi^{-1}(p_k), \quad k = 0, 1, 2, \ldots\} \tag{26}$$

can be represented in the form

$$Q = Q_1 \bigcup Q_2 \ ,$$

where

$$Q_1 = \{p_1 : p_1 = \pi^{-1}(p_0), \quad p_0 \in N\},$$

$$Q_2 = \bigcup_{p_1 \in P} \{p_{k+1} : p_{k+1} = \pi^{-1}(p_k), \quad k = 1, 2, 3, \ldots\}.$$

Q_1 coincides with P according to the Euclid theorem (1). On the other hand Q_2 is contained in P. Thus the right hand side of (26) coincides with P (i.e. $P \equiv Q$).

The set Q can be also represented in the form

$$P \equiv Q = Q_3 \bigcup Q_4 \ , \tag{27}$$

where

$$Q_3 = \bigcup_{p_0 \in N \setminus P} r_{p_0} \quad and \quad Q_4 = \bigcup_{p_0 \in P} r_{p_0} \ .$$

Owing to relation (15) from lemma 3 it follows that Q_4 appears to be a fraction of Q_3 ($Q_4 \subset Q_3$). Now it follows from (27) that $P \equiv Q_3$. []

3 Implications of the prime number separating theorem

Since $N \backslash P = C \bigcup \{1\} \equiv \overline{C}$ where C is the set of composite numbers it appears that PNST is mapping the elements of the extended set of composite numbers $p_0 \in \overline{C}$ into the infinite prime number rays r_{p_0}. So that we come to

Corollary 1 *There is an unique reciprocal mapping*

$$\varphi_1(p_0) : \overline{C} \longrightarrow \ {}^2P,$$

which maps the elements p_0 of the set $\overline{C}$ into the principal infinite prime number rays r_{p_0}.

Let us note that

$$ {}^2P = \{r_{p_0}\}_{p_0 \in N \backslash P} = \{p_{\mu\nu}\} \begin{array}{l} \mu = 1, 2, 3, \dots \\ \nu = 1, 2, 3, \dots \end{array}$$

denotes the prime number representation matrix introduced by eq. (25).

The corollary 1 on its turn is showing that an analogous matrix representation exists for the natural numbers too:

$$ {}^2N = \{\overline{C},\ {}^2P\}.$$

The upper left hand side of the infinite matrix ${}^2N$ is given in the Appendix.

Theorem 1 has diverse implications. Here we shall mention just one more. Let us denote the following classes of Eratosthenes rays:

$$\begin{array}{rcl}
K_1 & = & r_1, \\
K_2 & = & \{r_i\}_{i=2^j}\ ,\ j = 1, 2, 3, \dots, \\
K_3 & = & \{r_i\}_{i=3^j} \bigcup r_{2.3},\ j = 1, 2, 3, \dots, \\
K_5 & = & \{r_i\}_{i=5^j} \bigcup r_{2.5} \bigcup r_{3.5},\ j = 1, 2, 3, \dots, \\
 & \vdots & \\
K_p & = & \{r_i\}_{i=p^j} \bigcup \left(\bigcup\limits_{\alpha \le p, \alpha \in P} r_{\alpha.p} \right),\ j = 1, 2, 3, \dots\ .
\end{array}$$

Corollary 2 *There is an unique reciprocal mapping*

$$\varphi_2(p) : \overline{P} \longrightarrow \{K_p\}_{p \in \overline{P}},$$

which maps the elements of the set $\overline{P} = \{1\} \bigcup P$ *into the elements of the set of classes* $\{K_p\}_{p \in \overline{P}}$. *For the equivalence classes* K_p *the following set-theoretic equalities*

$$\bigcap_{p \in \overline{P}} K_p = \emptyset, \qquad \bigcup_{p \in \overline{P}} K_p = \ P.$$

take place.

The following two propositions for the elements of the matrix ${}^2P$ rows take place.

Theorem 2 *The series*

$$s(p_0) = \sum_{i=1}^{\infty} \frac{1}{p_i(p_0)}, \qquad p_0 \in N \backslash P$$

are convergent.
Proof

Following the inequalities (13) from lemma 2(2) the series $s(1)$ majorates all the series $s(p_0)$, for $p_0 \in C$. On its turn for $i \geq 5$ the series $s(1)$ is majorated by the series $\sum_{i=1}^{\infty} 1/i^2$. Indeed the inequalities (7), (9) and $5^2 < 31 = p_5(1)$, imply the following chain of inequalities

$$i^2 < 2i^2 < \pi^{-1}(i^2) < \pi^{-1}(p_{i-1}(1)), \quad for\ all\ i = 6, 7, 8, \dots \ .$$

[]

The estimate (8) from lemma 1 implies

Lemma 4 *The spacing between two adjacent elements of a ray from* $\{r_{p_0}\}_{p_0 \in \overline{C}}$ *except for* $p_{1,1} = 2 \in r_1$ *only is estimated by*

$$p_{(k+1)p_0} - p_{kp_0} > p_{kp_0}(\ln p_{kp_0} - 1), \quad for\ k = 1, 2, 3, \dots \ . \tag{28}$$

Theorem 1 and its implications are applicable to all mathematical constructions which include countable sets, or even sets containing finite segments of N or P only.

4 An analytic form of the prime number inner distribution law

The right hand side of relation (25) gives the definition for the recurrent element of any ray r_{p_0}, $p_0 \in \overline{C}$ in terms of the preceding element of the same ray only:

$$p_0 \in \overline{C}, \quad p_{j+1} = \pi^{-1}(p_j), \quad j = 0, 1, 2, 3, \dots \ . \tag{29}$$

This rule we comprehend to be the prime number inner distribution law — PNIDL.

The disadvantage of the law (29) is that the function $\pi^{-1}(x)$ (as well as the function $\pi(x)$) can be realized by means of the Eratosthenes sieve only. The Legendre formula (2) and its generalizations (see [8], p.343]) can not be used for the purpose.

So far the attempts to derive analytic formulae for $\pi^{-1}(p_j)$ for all $j \times p_0$ led to formulae which do not allow for a new information on prime numbers. All these formulae represent $\pi^{-1}(x)$ as a discrete function. Using these formulae it is only possible to determine these prime numbers which are initially presupposed by their construction. These formulae can not be extrapolated so as to increase the amount of prime numbers (formula (2) is an example of that type of formulae).

We shall show here that the Eratosthenes rays can be approximated by continuous functions which have extrapolation properties.

For an arbitrary row of the matrix ${}^2P$ from the Appendix we consider the solution of the quadratic system of $m = 2n$ equations

$$\left\{ \sum_{k=1}^{n} \alpha_{kp_0} q_{kp_0}(j) e^{-\beta_{kp_0} j} = \frac{\ln\ln\ln p_{1p_0}}{\ln\ln\ln p_{jp_0}} \right\}_{j=1,2,3,\ldots,m}, \tag{30}$$

respective to the n unknown pairs $\{\alpha_{kp_0}, \beta_{kp_0}\}_{k=1,2,3,\ldots,n}$, where

$$\begin{aligned} n &= 4 \;\; \textit{in case of} \;\; 1 \le p_0 \le 18 \;\; (\textit{the first } 11 \textit{ rays from Appendix}) \\ n &= 3 \;\; \textit{in case } 20 \le p_0 \le 64 \;\; (\textit{the succeeding } 32 \textit{ rays from Appendix}) \\ & \quad \textit{and} \\ n &= 2 \;\; \textit{in case of } 65 \le p_0 \le 132 \;\; (\textit{the last } 57 \textit{ rays from Appendix}) \end{aligned}$$

The polynomials q_{kp_0} in system (30) take the values

$$q_{kp_0}(j) \equiv 1 \;\; \textit{in case } 1 \le p_0 \le 16, \; 20 \le p_0 \le 28, \; 32 \le p_0 \le 132; \tag{31}$$

$$q_{1,18}(j) \equiv q_{2,18}(j) \equiv 1, \quad q_{3,18}(j) \equiv q_{4,18}(j) \equiv j \;\; \textit{for } p_0 = 18; \tag{32}$$

and

$$q_{1,30}(j) \equiv q_{2,30}(j) \equiv 1, \quad q_{3,30}(j) \equiv j, \;\; \textit{for } p_0 = 30. \tag{33}$$

Remark 1 *There are only two exceptions appearing for the first 100 rays from the Appendix for which $q_{kp_0}(j) \neq 1$. These are $p_0 = 18$ and $p_0 = 30$.*

Remark 2 *Constituting the system (30) for $p_0 = 1, 4, 6$ the first prime number p_{1p_0} is taken to be that for which the inequality $\ln\ln\ln(p_{1p_0}) > 0$ takes place for the first time. So that $p_{1,1} = 31$, $p_{1,4} = 17$, $p_{1,6} = 41$.*

The principle feature of the system (30) is that it is exactly soluble. The coefficients

$$\{\alpha_{kp_0}, \beta_{kp_0}\} \begin{array}{l} k = 1, 2, 3, \ldots, n \\ p_0 = 1, 4, 6, \ldots, 132 \end{array},$$

satisfy the system (30) with residuals $< 10^{-16}$. The solutions of the system (30) have also the following properties:

1. the amplitudes α_{kp} and the decrements β_{kp} are positive numbers;
2. α_{kp} and β_{kp} decrease when the index k is increasing;
3. the increase of the amplitudes α_{kp_0} corresponds to an increase of the decrements β_{kp_0} (this trend is strictly manifested for the first 26 rays);
4. each pair of two consequent elements of the ray r_{p_0} determine a new term of the sum (30).

The nonlinear systems (30) were analyzed by means of the program AFXY [9], which determines the number of solutions and their accuracy according to the method developed in [10] – [14]. It was established that all systems (30) – (31) are uniquely soluble, while the systems (30) – (32), (30) – (33) have triple solutions. In average the coefficients $\{\alpha_{kp_0}, \beta_{kp_0}\}$ have 8 – 10 correct decimal signs.

The solution of the system (30) led to the following approximation for the function $\pi^{-1}(x)$ which covers all the elements contained in the first 100 Eratosthenes rays from the Appendix:

$$\tilde{\pi}^{-1}(x; p_0) = \exp\exp\exp\frac{\ln\ln\ln p_{1p_0}}{\eta(x; p_0, n)}, \quad x \in [1, m] \subset R^1, \tag{34}$$

where

$$\eta(x; p_0, n) = \sum_{k=1}^{n} \alpha_{kp_0} q_{kp_0}(x) e^{-\beta_{kp_0} x}.$$

The formula

$$p_{jp_0} = \quad round - off \;\; (\tilde{\pi}^{-1}(j; p_0)) \tag{35}$$

exactly reproduces the prime numbers from the Appendix up to the even number $j^* \leq m$.

The function $\tilde{\pi}^{-1}(x)$ predicts the values of the new prime numbers p^f_{2n+1}. As seen from Table where the mean accuracy values in respect to p_0 are given

$$\delta_n = \frac{|p_{2n+1} - p^f_{2n+1}|100}{p_{2n+1}}$$

its prediction accuracy increases with the increase of the number n.

Table

n	2	3	4
δ_n	21% – 16%	5% – 1.7%	2.5% – 0.19%

To compare with we point out that for $p_0 = 4$ the relative accuracy of the solution $p^f_{9,4}$ obtained from the equation

$$li\ x = p_{jp_0} \tag{36}$$

when the right hand side is $p_{jp_0} = p_{8,4}$ equals to 0.004%

Let l is the number of the correct decimal signs (the length of a computer word) for which the arithmetic floating point operations in a given computing environment (the computer) are produced. The above found approximation for $\tilde{\pi}^{-1}(x)$, which is limited with respect to its domain of definition, suggests to check the hypothesis:

Conjecture 1 *For any ray r_{p_0}, $p_0 \in \overline{C}$ there exist finite numbers l and $n^*(l)$ such that the natural numbers $\tilde{p}_{mp_0}$ with $m = 2n^*(l) + 1$ predicted by formulae (34), (35) lye closer to the prime numbers p_{mp_0} than the solutions x of equation (36) when its right hand sides are $p_{(m-1)p_0}$.*

5 The set of origins of the Eratosthenes rays and coagulates of prime numbers

The set

$$P_1 = \left\{ p_1(p_0) : p_1(p_0) = \pi^{-1}(p_0), \quad p_0 \in \overline{C} \right\} = \{p_{\mu 1}\}_{\mu=1,2,3,\ldots}$$

consists from the start-points (origins) of the principal rays $\{r_{p_0}\}_{p_0 \in \overline{C}}$.

The set of prime numbers ${}^2P = \{p_{\mu\nu}\}_{\substack{\mu = 1,2,3,\ldots \\ \nu = 1,2,3,\ldots}}$ is separated according to PNST onto two new infinite subsets

$${}^2P = \{P_1, P_{erat}\}, \quad where \quad P_{erat} = \{p_{\mu\nu}\}_{\substack{\mu = 1,2,3,\ldots \\ \nu = 2,3,4,\ldots}} .$$

The set P_1 takes a special place amidst the prime numbers. It seems unlikely that a nonasymptotic distribution law similar to the PNIDL (29), accounting for the rows of the matrix P_{erat} too, can be derived for it. Studying the nonasymptotic

distribution of the elements of the subset P_1 amidst the prime numbers one can encounter its main peculiarity: P_1 contains all possible coagulates of prime numbers such as twin pairs and other closely disposed sequences of prime numbers.

Let the sets of twin-pairs, twin-triples, twin-quadruples, twin-quintuples of prime numbers greater than 5 are denoted respectively by

$$\begin{aligned} T_1 &= \{\overline{p}_i,\ \overline{p}_i+2\}_{i=1,2,3,\dots,l_1}\,, \\ T_2 &= \{\overline{p}_i,\ \overline{p}_i+2,\ \overline{p}_i+6\}_{i=1,2,3,\dots,l_2}\,, \\ T_3 &= \{\overline{p}_i,\ \overline{p}_i+2,\ \overline{p}_i+6,\ \overline{p}_i+8\}_{i=1,2,3,\dots,l_3}\,, \\ T_4 &= \{\overline{p}_i,\ \overline{p}_i+2,\ \overline{p}_i+6,\ \overline{p}_i+8,\ \overline{p}_i+12\}_{i=1,2,3,\dots,l_4}\,. \end{aligned}$$

Following from lemma 4 is the basic proposition which binds the sets $T_1,\ T_2,\ T_3$, and T_4 with the set P_1:

Theorem 3 *None of the elements of the subsets $T_1,\ T_2,\ T_3$ and T_4 is not contained in the Eratosthenes rays $\{r_{p0}\}_{p0} \in \overline{C}$. For any $i = 1, 2, 3, \dots, l_k,\ (k = 1, 2, 3, 4)$ only one element from pairs T_1, only two elements from the triplets T_2 and the quadruplets T_3, and only three elements from the quintuplets T_4 could not be origins of certain principal ray $\{r_{p0}\}_{p0} \in \overline{C}$.*

In theorem 3 the words "could not be origins" mean a probability whose amount among the first 104683 elements of the set T_1 does not exceed the value 0.15 (1270 twin-pairs are contained among these first elements). The elements of the sets $T_2,\ T_3$, and T_4 occur much more rarely amidst the natural numbers compared to these of the set T_1. Thus theorem 3 is showing that the essential part of the prime numbers included in the sets $T_1,\ T_2,\ T_3$ and T_4 appear to be origins of principal Eratosthenes rays.

A twin-pair assigns a characteristic "arhythmicity of condensation" among the coagulates from $T_2,\ T_3$ and T_4 as well as among all elements of the set P_1. Let us note that twin-pairs occur twice in T_3 and T_4. For instance in the elements $t_{4,1} = \{11, 13, 17, 19, 23\} \in T_4$ and $t_{4,2} = \{101, 103, 107, 109, 113\} \in T_4$ these are the pairs $\{11, 13\}, \{17, 19\}$, and $\{101, 103\},\ \{107, 109\}$ respectively.

Theorem 1 and theorem 3 together show that the Eratosthenes rays "coagulate between themselves" through the twin-pairs only and that according to the character of the "ray coagulates" these twin-pairs are classified into two new types:

1. pairs such as $\{\overline{p}_{iu}, \overline{p}_{iu}+2\}$ which simultaneously give origin of two new rays;
2. and the pairs $\{\overline{p}_{ib}, \overline{p}_{ib}+2\}$ for which one of the elements is the origin of a new ray while the other "coagulates a new ray "branching with an already existing ray.

Thus

$$T_1 = T_{1u} \bigcup T_{1b} \ , \tag{37}$$

where

$$T_{1u} = \{\overline{p}_{iu}, \overline{p}_{iu} + 2\}_{i=1,2,3,\ldots,l_5} \subset P_1$$

and

$$T_{1b} = \{\overline{p}_{ib}, \overline{p}_{ib} + 2\}_{i=1,2,3,\ldots,l_6} = T_1 \backslash T_{2u} \ .$$

As seen from the Appendix $\{71, 73\}, \{101, 103\}, \{137, 139\}$ and $\{149, 151\}$ are examples of u-pairs while $\{11, 13\}, \{17, 19\}, \{59, 61\}$ and $\{107, 109\}$ are examples of b-pairs.

The collective divisibility coefficient – CDC

$$D(l, s) = \frac{(d(l-s+1) + d(l-s+2) + \ldots + d(l))}{s+2}, \tag{38}$$

where $l \geq 13$, $s \geq 3$ are natural numbers, $d(\lambda)$ is the number of the divisors of the natural λ except the unit, when applied to the consequent triples ($s = 3$) of natural numbers p, $p+1$, $p+2$ leads to a new class of twin-pairs — those for which $D(l, 3) = 1$ (the Strongly Related Twins – T_{sr}):

$$\begin{aligned} T_{sr} &= \{\{11, 13\}, \{17, 19\}, \{29, 31\}, \{41, 43\}, \{101, 103\}, \{137, 139\}, \ldots\} = \\ &\quad \{\overline{\overline{p}}_i, \overline{\overline{p}}_i + 2\}_{i=1,2,3,\ldots,l_7} \ . \end{aligned} \tag{39}$$

The even number $p+1$ from the twin-triple $\{p, p+1, p+2\}$ is factorized as follows

$$p + 1 = 2.3.\sigma, \quad \sigma \in P. \tag{40}$$

The numbers $\sigma = 2, 3, 5, 7, 17, 23$ from the factorization (40) correspond to the twin-pairs (39) and form the origin of a new subset $\Sigma \subset P$ which has a curious property: the last decimal digit of all $\sigma \in \Sigma$ is either 3 or 7 (never 1 or 9; this property is checked up to the T_{sr}–pair $\{47777, 47779\}$).

Remark 3

$$\min_{l \geq 13, s \geq 3} D(l, s) = D(l, 3) = 1 \ .$$

Remark 4 *The exceptions mentioned in Remark 1 are the even numbers 18 and 30 corresponed to* T_{sr} $\{17, 19\}$ *and* $\{29, 31\}$.

The decomposition (39) is carried over the elements of the set T_{sr} too

$$T_{sr} = T_{(sr)u} \bigcup T_{(sr)b} \ ,$$

where

$$T_{(sr)u} = \{\overline{\overline{p}}_{iu},\ \overline{\overline{p}}_{iu} + 2\}_{i=1,2,3,\ldots,l_8}$$

and

$$T_{(sr)b} = \{\overline{\overline{p}}_{ib},\ \overline{\overline{p}}_{ib} + 2\}_{i=1,2,3,\ldots,l_9} \ .$$

The existence of a potential infinity $l_k = \infty$ for the lengths $l_k,\ k = 1, 2, 3, \ldots, 9$ of the introduced sets of prime numbers is still not proved even for the simplest case $k = 1$.

The following extension of the V. Brun [15] theorem is possible:

Theorem 4 *If $l_k = \infty$ for $k = 1, 5, 6, 7, 8, 9$ the series*

$$\sum_{i=1}^{\infty} \left(\frac{1}{v_i} + \frac{1}{v_i + 2} \right), \quad \textit{where } v_i = \overline{p}_i,\ \overline{p}_{iu},\ \overline{p}_{ib},\ \overline{\overline{p}}_i,\ \overline{\overline{p}}_{iu},\ \overline{\overline{p}}_{ib}$$

are convergent.

In case of $l_2 = l_3 = l_4 = \infty$ a statement analogous to theorem 4 will take place for the elements of the sets T_2, T_3, and T_4.

Besides the sets T_2 and T_3 one can consider the sets of coagulates of the type $\{p_i, p_i + 4, p_i + 6\},\ \{p_i, p_i + 2, p_i + 8\},\ \{p_i, p_i + 4, p_i + 6, p_i + 10\}$ and $\{p_i, p_i + 2, p_i + 8, p_i + 12\}$ too. For these sets the analogues of theorems 3 and 4 are also correct.

Let us introduce the set

$$\tilde{P}_1 = P_1 \backslash \left(T_1 \bigcup T_2 \bigcup T_3 \bigcup T_4 \right) \equiv \{\tilde{p}_{i1}\}_{i=1,2,3,\ldots} \ .$$

Treating numerically the origins of the partial sums of the reciprocal values of the elements of the columns of the matrix ${}^2P$ we come to a hypothesis concerning the set of origins of the principal rays of P_1 which is opposite in since of the V. Brun theorem:

Conjecture 2 *The series*

$$\sum_{i=1}^{\infty} \frac{1}{w_i}, \quad \textit{where } w_i = p_{i1},\ \tilde{p}_{i1}, \quad p_{i\nu} \quad (\nu = 2, 3, 4, \ldots) \tag{41}$$

is divergent.

Remark 5 *The divergency of the series (41) is very slow for* $w_i = p_{i1},\ \tilde{p}_{i1}$ *and even more slow for* $w_i = p_{i\nu}$*. What is more the slowness is growing with* $\nu \to \infty$ *(?).*

The coagulates of primes elements of the sets T_1, T_2, T_3 and T_4 considered so far should be generalized as

$$coag\left(p, \{u_i\}_{i=1,2,3,\ldots,l}\right) = \{p, p+u_1, p+u_2, \ldots, p+u_l\}$$

where the steps $\{u_i\}$ should be smaller than the steps $p_{(k+1)p_0} - p_{kp_0} = \Delta\epsilon_{kp_0}$ in the nearest rays while the rays and their steps themselves (i.e. the particular p_0 and k) are determined by the initial prime number p and the length l of the generalized coagulate. The pointed out relativity of the values of p, l and $\Delta\epsilon_{kp_0}$ in $coag\left\{p, \{u_i\}_{i=1,2,\ldots,l}\right\}$ is used in order to obtain the conditions under which the set of generalized *coags* satisfy theorems analogous to theorems 3 and 4.

The general meaning of theorems 3 and 4, and conjecture 2 is that the main information on the originality of the nonasymptotic prime-numbers distribution is concentrated in the set of start–points P_1; by its nature the set P_1 contains "enough amount" of prime-numbers and they are disposed "enough closely" (conjecture 2), however only the set of the type of T_1, T_2, T_3 and T_4, which contain "relatively small amount of elements" (theorem 4) introduce the characteristic, unique arrhythmia of concentration of elements of the set P_1.

section*Appendix

The left upper corner of the infinite matrix ^2N.

1	2	3	5	11	31
	127	709	5381	52711	648391
	9737333	174440041...			
4	7	17	59	277	1787
	15299	167449	2269733	37139213	718064159...
6	13	41	179	1063	8527
	87803	1128889	17624813	326851121...	
8	19	67	331	2221	19577
	219613	3042161	50728129...		
9	23	83	431	3001	27457
	318211	4535189	77557187...		
10	29	109	599	4397	42043
	506683	7474967	131807699...		
12	37	157	919	7193	72727
	919913	14161729	259336153...		
14	43	191	1153	9319	96797
	1254739	19734581	368345293...		
15	47	211	1297	10631	112129
	1471343	23391799	440817757...		
16	53	241	1523	12763	137077
	1828669	29499439	563167303...		
18	61	283	1847	15823	173867
	2364361	38790341	751783477...		
20	71	353	2381	21179	239489
	3338989	56011909...			
21	73	367	2477	22093	250751
	3509299	59053067...			
22	79	401	2749	24859	285191
	4030889	68425619...			
24	89	461	3259	30133	352007
	5054303	87019979...			
25	97	509	3637	33967	401519
	5823667	101146501...			
26	101	547	3943	37217	443419
	6478961	113256643...			

The left upper . . . Continuation

27	103 563 4091 38833 464939 6816631 119535373...
28	107 587 4273 40819 490643 7220981...
30	113 617 4549 43651 527623 7807321...
32	131 739 5623 55351 683873 10311439...
33	137 773 5869 57943 718807 10875147...
34	139 797 6113 60647 755387 11469013...
35	149 859 6661 66851 839483 12838937...
36	151 877 6823 68639 864013 13243033...
38	163 967 7607 77431 985151 15239333...
39	167 991 7841 80071 1021271 15837299...
40	173 1031 8221 84347 1080923 16827557...
42	181 1087 8719 90023 1159901 18143603...
44	193 1171 9461 98519 1278779 20137253...
45	197 1201 9739 101701 1323503 20890789...
46	199 1217 9859 103069 1342907 21219089...
48	223 1409 11743 125113 1656649 26548261...
49	227 1433 11953 127643 1693031 27170047...
50	229 1447 12097 129229 1715761 27560453...
51	233 1471 12301 131707 1751411 28171007...

The left upper . . . Continuation

52	239	1499	12547	134597	1793237
	28889363...				
54	251	1597	13469	145547	1950629
	31599859...				
55	257	1621	13709	148439	1993039
	32332763...				
56	263	1669	14177	153877	2071583
	33691309...				
57	269	1723	14723	160483	2167937
	35368547...				
58	271	1741	14867	162257	2193689
	35815873...				
60	281	1823	15641	171697	2332537
	38235377...				
62	293	1913	16519	182261	2487943...
63	307	2027	17627	195677	2685911...
64	311	2063	17987	200017	2750357...
65	313	2081	18149	202001	2779781...
66	317	2099	18311	204067	2810191...
68	337	2269	20063	225503	3129913...
69	347	2341	20773	234293	3260657...
70	349	2351	20899	235891	3284657...
72	359	2417	21529	243781	3403457...
74	373	2549	22811	259657	3643579...
75	379	2609	23431	267439	3760921...
76	383	2647	23801	271939	3829223...
77	389	2683	24107	275837	3888551...
78	397	2719	24509	280913	3965483...
80	409	2803	25423	292489	4142053...
81	419	2897	26371	304553	4326473...
82	421	2909	26489	305999	4348681...
84	433	3019	27689	321017	4578163...
85	439	3067	28109	326203	4658099...
86	443	3109	28573	332099	4748047...
87	449	3169	29153	339601	4863959...
88	457	3229	29803	347849	4989697...
90	463	3299	30557	357473	5138719...

The left upper . . . End.

91	467	3319	30781	360293	5182717...
92	479	3407	31667	371981	5363167...
93	487	3469	32341	380557	5496349...
94	491	3517	32797	386401	5587537...
95	499	3559	33203	391711	5670851...
96	503	3593	33569	396269	5741453...
98	521	3733	35023	415253	6037513...
99	523	3761	35311	418961	6095731...
100	541	3911	36887	439357	6415081...
102	557	4027	38153	455849	6673993...
104	569	4133	39239	470207	6898807...
105	571	4153	39451	472837	6940103...
106	577	4217	40151	481847	7081709...
108	593	4339	41491	499403	7359427...
110	601	4421	42293	510031	7528669...
111	607	4463	42697	515401	7612799...
112	613	4517	43283	522829	7730539...
114	619	4567	43889	530773	7856939...
115	631	4663	44879	543967	8066533...
116	641	4759	45971	558643	8300687...
117	643	4787	46279	562711	8365481...
118	647	4801	46451	565069	8402833...
119	653	4877	47297	576203	8580151...
120	659	4933	47857	583523	8696917...
121	661	4943	47963	584999	8720227...
122	673	5021	48821	596243	8900383...
123	677	5059	49207	601397	8982923...
124	683	5107	49739	608459	9096533...
125	691	5189	50591	619739	9276991...
126	701	5281	51599	633467	9498161...
128	719	5441	53353	657121	9878657...
129	727	5503	54013	665843...	
130	733	5557	54601	673793...	
132	743	5651	55681	688249...	
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